Some ways to calculate the characteristic function of the Cauchy distribution

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Chapter 1

Preliminaries

1.1 Introduction

Our ultimate goal of this document is to introduce some ways to calculate the integral

$$I = I(\alpha) = \int_0^\infty \frac{\cos \alpha x}{x^2+1}\,dx, \qquad (\alpha \in \mathbb{R}) \tag{1.1}$$

which often arises in some areas of mathematics. The characteristic function of the standard Cauchy distribution is a good example where we are led to evaluate (1.1).

There are various ways to achieve this goal known so far, amid which the method of contour integration is most famous. But often they rely on some advance theories or cumbersome justifications, thus it would be timely to review some materials that we need hereafter. Readers who are familiar to these concepts may skip these supplementary sections.

1.2 A Brief Review on Real Analysis

The principal purpose of utilizing real analysis techniques here is to provide a theoretical background for advanced calculus. This is epitomized by a set of theorems that allows us to interchange various limiting operators, Leibniz's integral rule for instance. So we focus on particular results that have direct applications to this document, rather than self-contained reviews of a theory.

1.2.1 Uniform Convergence

You can refer to [3] for details and proofs appearing this subsection. This book is a gentle, yet powerful and comprehensive introduction to analysis.

Definition 1. *Let $A \subset \mathbb{R}^m$ and $f_k : A \to \mathbb{R}^n$ be a sequence of functions such that for every $\varepsilon > 0$, there is a positive integer L such that $|f_k(x) - f(x)| < \varepsilon$ for all $x \in A$ and $k \geq L$. Under these conditions, we say that (f_k)* ***converges uniformly to f on*** *A.*

We will assume that f_k and f are functions from $A \subset \mathbb{R}^m$ to $\mathbb{R}^n$ throughout this subsection, unless stated otherwise.

Theorem 2. *Let (f_k) be a sequence of continuous functions. If (f_k) converges uniformly to f, then f is also continuous.*

Theorem 3 (Weierstrass M-test)**.** *Suppose there is a sequence of non-negative numbers (M_k) such that $|f_k(x)| \leq M_k$ for all $x \in A$ and $k = 1, 2, 3, \cdots$, and $\sum_{k=1}^{\infty} M_k < +\infty$. Then $\sum_{k=1}^{\infty} f_k$ converges uniformly and absolutely.*

Theorem 4. *Let $f_k : [a,b] \to \mathbb{R}$ be a sequence of Riemann integrable functions, and converges uniformly to $f : [a,b] \to \mathbb{R}$ on $[a,b]$. Then f is Riemann integrable on $[a,b]$, and*

$$\lim_{k\to\infty} \int_a^b f_k(x)\, dx = \int_a^b f(x)\, dx.$$

Theorem 5. *Let $f_k : (a,b) \to \mathbb{R}$ be a sequence of C^1 functions. Suppose (f_k) converges pointwise to $f : (a,b) \to \mathbb{R}$ and (f_k') converges uniformly to $g : (a,b) \to \mathbb{R}$. Then f is differentiable and $f' = g$.*

1.2.2 Lebesgue Integration

Materials from this subsection entirely rely on [2]. This book will serve as an excellent introduction to measure theory and elementary functional analysis.

Theorem 6 (Monotone Convergence Theorem)**.** *Let (f_k) be a sequence of non-negative measurable functions such that $f_k(x) \nearrow f(x)$ a.e. x. then*

$$\lim_{n\to\infty} \int f_n = \int f.$$

Theorem 7 (Dominated Convergence Theorem)**.** *Let (f_k) be a sequence of measurable functions such that $f_k(x) \to f(x)$ a.e. x as $k \to \infty$. If there is an integrable function g such that $|f_k(x)| \le g(x)$ a.e. x for all k, then*

$$\lim_{n\to\infty} \int f_n = \int f.$$

Theorem 8 (Fubini's Theorem)**.** *Suppose $f(x,y)$ is either integrable or non-negative measurable*[1] *on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then*

$$\int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} f(x,y)\, dydx = \int f = \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} f(x,y)\, dxdy.$$

1.3 A Brief Review on Complex Analysis

Most of this subsection is based on [1].

Theorem 9 (Cauchy Integration Theorem)**.** *Let f be a holomorphic function defined on an open set Ω, and γ be a piecewise C^1 simple closed curve which is counter-clockwise oriented such that the closure of its interior is contained in Ω. Then for any z which lies in the interior of γ,*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

[1] This condition is formulated by Tonelli, so Fubini's theorem restricted to this condition is often called Tonelli's theorem.

Chapter 2

Calculations

2.1 Method of Contour Integration

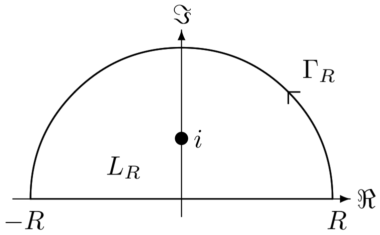
This solution adopts the method of contour integration, the standard method of calculating contour integration.

By symmetry, we have

$$2I = \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2+1}\,dx.$$

Thus it suffices to evaluate the rightmost integral. We assume $\alpha > 0$ without loss of generality.

Let $R > 1$ and C_R denote the counter-clockwised contour consisting of the line segment L_R from $-R$ to R and the upper semicircular arc Γ_R of radius R centered at the origin.



Then we have

$$\int_{C_R} \frac{e^{i\alpha z}}{z^2+1}\,dz = \int_{-R}^{R} \frac{e^{i\alpha z}}{z^2+1}\,dz + \int_{\Gamma_R} \frac{e^{i\alpha z}}{z^2+1}\,dz =: I_R + J_R. \tag{2.1}$$

Now Cauchy integration theorem shows that the leftmost hand side of (2.1) is equal to

$$\int_{C_R} \frac{e^{i\alpha z}}{z^2+1}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{i\alpha z}}{z^2+1} = \pi e^{-\alpha}.$$

On the other hand, $I_R \to 2I$ as $R \to \infty$. So it remains to evaluate the limit of J_R. We claim that this vanishes as $R \to \infty$, hence by (2.1),

$$2I = \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2+1}\,dx = \pi e^{-\alpha}.$$

Indeed,

$$|J_R| \le \int_{\Gamma_R} \left| \frac{e^{i\alpha z}}{z^2+1} \right| |dz| \le \int_{\Gamma_R} \frac{1}{R^2-1}\,|dz| = \frac{\pi R}{R^2-1}$$

and letting $R \to \infty$ proves the claim.

2.2 Method using Real Analysis - I

We introduce a simple lemma:

Lemma 10. *Suppose both $f(x)$ and $xf(x)$ are integrable on $[0,\infty)$, and $g(x)$ is a differentiable function such that both $g(x)$ and $g'(x)$ is bounded on $[0,\infty)$. Then*

$$F(\alpha) = \int_0^\infty f(x)g(\alpha x)\,dx$$

is continuous on $[0,\infty)$, differentiable on $(0,\infty)$ and

$$F'(\alpha) = \int_0^\infty xf(x)g'(\alpha x)\,dx.$$

Proof of lemma. Let $M > 0$ be a bound for $g'(x)$. Then for $\alpha > 0$ and $0 < |h| \ll \alpha$, mean value theorem applied to

$$\frac{F(\alpha+h) - F(\alpha)}{h} = \int_0^\infty f(x)\,\frac{g(\alpha x + hx) - g(\alpha x)}{h}\,dx$$

shows that the integrand is dominated by an integrable function $Mxf(x)$, thus dominated convergence theorem shows that we can interchange the limit as $h \to 0$ and the integration. This gives

$$\begin{aligned}\lim_{h\to 0}\frac{F(\alpha+h) - F(\alpha)}{h} &= \int_0^\infty f(x)\left(\lim_{h\to 0}\frac{g(\alpha x + hx) - g(\alpha x)}{h}\right)dx \\ &= \int_0^\infty xf(x)g'(\alpha x)\,dx,\end{aligned}$$

proving the lemma.

Now we return to the original problem. As before, we assume $\alpha \geq 0$. By integration by parts,

$$\begin{aligned}I(\alpha) &= \frac{\sin\alpha x}{\alpha}\frac{1}{x^2+1}\bigg|_0^\infty - \int_0^\infty \frac{\sin\alpha x}{\alpha}\left(-\frac{2x}{(x^2+1)^2}\right)dx \\ &= \frac{1}{\alpha}\int_0^\infty \frac{2x\sin\alpha x}{(x^2+1)^2}\,dx.\end{aligned}$$

Then applying lemma 10 to $\alpha I(\alpha)$, we have

$$\alpha I'(\alpha) + I(\alpha) = \int_0^\infty \frac{2x^2\cos\alpha x}{(x^2+1)^2}\,dx = 2I(\alpha) - \int_0^\infty \frac{2\cos\alpha x}{(x^2+1)^2}\,dx,$$

or equivalently

$$\alpha I'(\alpha) - I(\alpha) = -\int_0^\infty \frac{2\cos\alpha x}{(x^2+1)^2}\,dx.$$

Differentiating with aid of lemma 10 again,

$$\alpha I''(\alpha) = \int_0^\infty \frac{2x\sin\alpha x}{(x^2+1)^2}\,dx = \alpha I(\alpha).$$

Therefore $I''(\alpha) - I(\alpha) = 0$. Then standard theory of linear differential equation shows that the general solution has the form

$$I(\alpha) = Ae^{\alpha} + Be^{-\alpha} \tag{2.2}$$

for some constants A, B. So it remains to determine A and B. Since $I(\alpha)$ is bounded, A must vanish. Then (2.2) reduces to

$$I(\alpha) = Be^{-\alpha},$$

showing that $B = I(0) = \frac{\pi}{2}$. Therefore we have

$$I(\alpha) = \frac{\pi}{2}e^{-\alpha}.$$

Bibliography

[1] Rami Shakarchi Elias M. Stein. *Complex Analysis*. Princeton University Press, 2003.

[2] Rami Shakarchi Elias M. Stein. *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*. Princeton University Press, 2005.

[3] Michael J. Hoffman Jerrold E. Marsden. *Elementary Classical Analysis*. Freeman, 2nd edition, 1993.